

A GENERALIZATION OF TOKUYAMA'S FORMULA TO THE HALL-LITTLEWOOD POLYNOMIALS

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ABSTRACT. A theorem due to Tokuyama expresses Schur polynomials in terms of statistics from Gelfand-Tsetlin patterns, providing a deformation for the Weyl character formula and two other classical results: Stanley's formula and Gelfand's parametrization. The Hall-Littlewood polynomials are a deformation of several classes of symmetric polynomials, including the Schur polynomials. We generalize Tokuyama's formula to the Hall-Littlewood polynomials by extending Tokuyama's statistics, yielding an expression for the Hall-Littlewood polynomials as a generating function of Gelfand-Tsetlin patterns.

1. INTRODUCTION

Symmetric polynomials have proven an invaluable tool in the study of group representations. A particularly important class is the Schur polynomials, a ring of polynomials indexed by partitions which serve as a \mathbb{Z} -basis for the graded ring of symmetric functions Λ [Mac88]. Schur functions, computed via the Weyl character formula, encode the characters of irreducible representations of general linear groups. Tokuyama gave a deformation of the Weyl character formula for $GL(n, \mathbb{C})$. His formula expresses Schur polynomials in terms of statistics obtained from strict Gelfand-Tsetlin (GT) patterns and includes several other classical results as specializations. One is the Gelfand parametrization of the weight vectors of general linear groups (see [GZ50]); the other is Stanley's formula for the Schur q -functions (see [Sta86], [Tok88] for more detail).

Though Tokuyama's formula has been extended to other types of groups, e.g. in [HK02], analogous results for other classes of symmetric functions remain comparatively unexplored. The Hall-Littlewood polynomials are another class of symmetric functions which generalize the Schur polynomials through a deformation along a parameter t [Mac79, Chapter III], making them natural to consider in generalizing Tokuyama's formula. As well as forming a $\mathbb{Z}[t]$ -basis of Λ which interpolates between the dual bases of the Schur functions and the monomial symmetric functions at $t = 0$ and $t = 1$ respectively, these polynomials are used to determine characters of Chevalley groups over local and finite fields [Tok88]. Stanley's formula, a specialization of Tokuyama's, expresses the Hall-Littlewood polynomials at the singular value $t = -1$ (commonly known as the Schur q -functions, [Mac79, Chapter III]) as a summation over strict GT patterns. However, in existing literature there does not exist an analogue of Tokuyama's formula capable of expressing the general Hall-Littlewood polynomials as a summation over combinatorial data from Gelfand-Tsetlin patterns.

In this paper, we provide such a deformation. This formula, in addition to linking the classical specializations of Tokuyama, also includes the formulas for Schur q -functions and the monomial symmetric functions as specializations.

Section 2 introduces the Schur and Hall-Littlewood polynomials as well as Tokuyama's result. Section 3 defines a set of raising operators and presents a related proposition. In Section 4 we extend Tokuyama's combinatorial notations on Gelfand Tsetlin patterns and

state the main theorem of this paper; Section 5 presents the proof of this theorem. We also include a computed example of a case of this theorem for clarity in the Appendix. Section 6 provides an alternate statement of this main result. We conclude in Section 7 with interesting consequences and specializations of the results.

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2. PRELIMINARY NOTATION AND A THEOREM DUE TO TOKUYAMA

A partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ is a finite tuple of nonnegative integers. Unless otherwise stated, a partition will be assumed to be weakly decreasing with $\lambda_i \geq \lambda_{i+1}$ for all i . The *length* of a partition is the number of entries in the tuple and the size is $|\lambda| = \sum_{i=1}^n \lambda_i$. Partitions of equal length are added component-wise, and given two partitions λ and μ of lengths n and m respectively, we express their concatenation as $\lambda \parallel \mu = (\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_m)$.

A partition of particular importance is

$$(2.1) \quad \rho_n = (n-1, n-2, \dots, 1, 0).$$

We often drop the subscript n of ρ_n when its value is clear from context.

Let the polynomial $f(x)$ be short for $f(x_1, \dots, x_n)$ and $x^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \dots x_n^{\lambda_n}$. An element σ of the symmetric group S_n acts on a polynomial by permuting the variables x_i and on a partition by permuting the parts λ_i . We use $\text{sgn } \sigma$ as the standard sign function.

We denote the Weyl denominator as Δ_n given by

$$(2.2) \quad \Delta_n = \prod_{1 \leq i < j \leq n} (x_i - x_j),$$

and define a deformation $\Delta_n(t)$ by

$$(2.3) \quad \Delta_n(t) = \prod_{1 \leq i < j \leq n} (x_i - tx_j).$$

Note that $\Delta_n(1) = \Delta_n$ and $\Delta_n(0) = x^\rho$.

Theorem 2.1. By the Weyl character formula, the Schur polynomial corresponding to the partition λ of length n is

$$(2.4) \quad s_\lambda(x) = \sum_{\sigma \in S_n} \text{sgn } \sigma \cdot \frac{\sigma(x^{\lambda+\rho})}{\Delta_n}.$$

We define the Hall-Littlewood polynomials below, noting that it differs from the original definition given in [Mac79] by the omission of a stabilizing factor.

Definition 1. The Hall-Littlewood polynomial for a partition λ of length n is

$$(2.5) \quad R_\lambda(x; t) = \sum_{\sigma \in S_n} \sigma \left(x^\lambda \frac{\Delta_n(t)}{\Delta_n} \right)$$

Note: The traditional Hall-Littlewood polynomials defined in [Mac79] are

$$(2.6) \quad P_\lambda(x; t) = v_\lambda(t) R_\lambda(x; t),$$

for some suitable stabilizing factor $v_\lambda(t)$. Since the stabilizing factor may simply be adjoined to our results if necessary, we choose to omit it for the purpose of this paper, and refer to the polynomials $R_\lambda(x; t)$ as the Hall-Littlewood polynomials.

It is not difficult to see that $R_\lambda(x; 0) = s_\lambda(x)$ and $R_\lambda(x; 1) = \sum_{\sigma \in S_k} \sigma(x^\lambda)$, the latter being the monomial symmetric function $m_\lambda(x)$ without its stabilizer.

Definition 2. A Gelfand-Tsetlin (GT) pattern is a triangular array of nonnegative integers of the form:

$$\begin{array}{ccccccccc} a_{1,1} & & a_{1,2} & & a_{1,3} & & \dots & & a_{1,n} \\ & a_{2,2} & & a_{2,3} & & \dots & & & a_{2,n} \\ & & \dots & & \dots & & \dots & & \\ & & & a_{n-1,n-1} & & a_{n-1,n} & & & \\ & & & & a_{n,n} & & & & \end{array}$$

where each row is weakly decreasing and each $a_{i,j}$ for $i > 1$ satisfies

$$(2.7) \quad a_{i-1,j-1} \geq a_{i,j} \geq a_{i-1,j}.$$

We take $G(\lambda)$ to be the set of all GT patterns of top row λ . A *strict* GT pattern is one in which each row is strictly decreasing; we take $SG(\lambda) \subseteq G(\lambda)$ to be the set of all strict GT patterns with top row λ .

Furthermore, for some GT pattern, we define the tuple

$$(2.8) \quad r_i = (a_{i,i}, a_{i,i+1}, \dots, a_{i,n}),$$

which can be viewed as the i th row of the pattern expressed as a tuple.

We introduce Tokuyama's formula by presenting several statistics on entries in a GT pattern used in [Tok88].

Definition 3. An entry $a_{i,j}$ in a GT pattern is

- left-leaning if $a_{i,j} = a_{i-1,j-1}$,
- right-leaning if $a_{i,j} = a_{i-1,j}$, and
- special if it is neither left-leaning nor right-leaning.

The quantities $l(T)$, $r(T)$ and $s(T)$ denote the numbers of left-leaning, right-leaning and special entries in a GT pattern respectively.

For a GT pattern with n rows, the statistic $m_i(T)$ is defined as

$$(2.9) \quad m_i(T) = \begin{cases} |r_i| - |r_{i+1}| & \text{for } 1 \leq i \leq n-1 \\ |r_i| & \text{for } i = n \end{cases},$$

and consequently the n -tuple $m(T)$ is taken to be

$$(2.10) \quad m(T) = (m_1(T), \dots, m_n(T)).$$

We are now able to state a theorem due to Tokuyama [Tok88], which expresses the Schur polynomials in terms of statistics from Gelfand-Tsetlin Patterns:

Theorem 2.2 ([Tok88]). For any decreasing partition λ of length n , we have

$$(2.11) \quad \Delta_n(q) \cdot s_\lambda(x) = \sum_{T \in SG(\lambda + \rho)} (1 - q)^{s(T)} (-q)^{l(T)} x^{m(T)}.$$

3. RAISING OPERATORS

In this section we present and prove an important proposition required for the proof of this paper's results. This proposition expresses certain Hall-Littlewood polynomials corresponding to increasing partitions as Hall-Littlewood polynomials of weakly decreasing partitions by using raising operators [Mac79, Chapter I].

Definition 4. A raising operator ϕ is a product of operations $[i\ j]$ with $i \leq j$ acting on some $\lambda \in \mathbb{Z}^n$ such that

$$(3.1) \quad [i\ j] \cdot (\lambda_1, \dots, \lambda_n) = (\lambda_1, \dots, \lambda_i - 1, \dots, \lambda_j + 1, \dots, \lambda_n).$$

The length of a raising operator ϕ , denoted $l(\phi)$, is defined as the number of operators in the minimal decomposition of ϕ into elementary operators of the form $[i\ i+1]$. The identity raising operator Id is assigned length zero.

Definition 5. Given a partition λ of length n , we recursively define $\Omega(\lambda)$ to be the set of raising operators such that

- The identity raising operator $\text{Id} \in \Omega(\lambda)$, and
- for all raising operators $\phi \in \Omega(\lambda)$, if the partition $\phi(\lambda) = (\lambda'_1, \dots, \lambda'_n)$ contains consecutive parts λ'_i and λ'_{i+1} such that $\lambda'_i = \lambda'_{i+1} + 2$, then $[i\ i+1] \cdot \phi \in \Omega(\lambda)$.

Example 3.1. For the partition $\lambda = (6, 4, 3, 1)$, we have the set

$$\Omega(\lambda) = \{\text{Id}, [1\ 2], [1\ 3], [2\ 4], [3\ 4], [1\ 2][3\ 4]\}.$$

With these notions, we demonstrate an interesting quality of $\Omega(\lambda)$ used in connection with the Hall-Littlewood polynomials which proves useful in Theorem 4.1.

Proposition 3.1. For all $\phi \in \Omega(\lambda + \rho)$, we have

$$(3.2) \quad R_{\phi(\lambda)}(x; t) = t^{l(\phi)} \cdot R_\lambda(x; t).$$

Proof. Let $\alpha = \lambda + \rho$. It is clear that 3.2 holds for $\phi = \text{Id}$. Therefore, due to the recursive definition of $\Omega(\alpha)$, it suffices to prove that for any elementary operator $[i\ i+1]$ acting upon consecutive parts α_i and α_{i+1} in the tuple α such that $\alpha_i = \alpha_{i+1} + 2$, or equivalently $\lambda_i = \lambda_{i+1} + 1$, we have

$$(3.3) \quad R_{[i\ i+1](\lambda)}(x; t) = t \cdot R_\lambda(x; t).$$

We now prove (3.3). Let $g(x) = \Delta_n(t)/(x_i - tx_{i+1})$ and $f(x) = (x_1^{\lambda_1} \cdots x_{i-1}^{\lambda_{i-1}}) \cdot (x_{i+2}^{\lambda_{i+2}} \cdots x_n^{\lambda_n})$, noting that both $g(x)$ and $f(x)$ are unchanged by the permutation $(i\ i+1)$. Then, setting $\lambda_i = a + 1$ and $\lambda_{i+1} = a$, we see that

$$\begin{aligned} & \sum_{\sigma \in S_n} (\text{sgn } \sigma) \sigma(x^\lambda \Delta_n(t)) \\ &= \sum_{\sigma \in S_n} (\text{sgn } \sigma) \sigma(f(x)g(x)x_i^{a+2}x_{i+1}^a) - \sum_{\sigma \in S_n} (\text{sgn } \sigma) \sigma(tf(x)g(x)x_i^{a+1}x_{i+1}^{a+1}). \end{aligned}$$

We know that if for some polynomial $h(x)$ there exists a transposition $(i\ j) \in S_n$ such that $(i\ j)(h(x)) = h(x)$, then the identity $\sum_{\sigma \in S_n} (\text{sgn } \sigma) \sigma(h(x)) = 0$ holds. Therefore, since $tf(x)g(x)x_i^{a+1}x_{i+1}^{a+1}$ is unchanged by the permutation $(i\ i+1)$, we have

$$(3.4) \quad \sum_{\sigma \in S_n} (\text{sgn } \sigma) \sigma(x^\lambda \Delta_n(t)) = \sum_{\sigma \in S_n} (\text{sgn } \sigma) \sigma(f(x)g(x)x_i^{a+2}x_{i+1}^a).$$

Similarly, we can find that

$$(3.5) \quad \sum_{\sigma \in S_n} (\text{sgn } \sigma) \sigma(x^{[i\ i+1](\lambda)} \Delta_n(t)) = (-t) \cdot \sum_{\sigma \in S_n} (\text{sgn } \sigma) \sigma(f(x)g(x)x_i^a x_{i+1}^{a+2}).$$

Replacing the permutations σ with $\sigma(i\ i+1)$ in the right hand side of (3.5) returns the right hand side of (3.4) multiplied by t . Thus, substituting this with the left hand side of (3.4) returns

$$\sum_{\sigma \in S_n} (\text{sgn } \sigma) \sigma(x^{[i\ i+1](\lambda)} \Delta_n(t)) = t \cdot \sum_{\sigma \in S_n} (\text{sgn } \sigma) \sigma(x^\lambda \Delta_n(t)).$$

Dividing through by Δ_n gives us the desired result. \square

4. STATISTICS ON GELFAND-TSETLIN PATTERNS AND MAIN THEOREM

To state the results of this paper, we require some notation. Instead of labelling each entry as left-leaning, right-leaning or special as in Definition 3, we label each entry with both a *left-sided* property and a *right-sided* property. The left-sided property encodes the relationship the entry $a_{i,j}$ has to the entry directly above it and to its left, namely $a_{i-1,j-1}$. Similarly, the right-sided property encodes the relationship that $a_{i,j}$ has to $a_{i-1,j}$.

The left-sided properties of a entry $p_l(a_{i,j})$ are assigned as

$$(4.1) \quad p_l(a_{i,j}) = \begin{cases} l \text{ (left)} & \text{if } a_{i,j} = a_{i-1,j-1} \\ al \text{ (almost-left)} & \text{if } a_{i,j} = a_{i-1,j-1} - 1, \\ s \text{ (special)} & \text{otherwise} \end{cases}$$

and similarly, the right-sided properties of an entry $p_r(a_{i,j})$ are assigned as

$$(4.2) \quad p_r(a_{i,j}) = \begin{cases} r \text{ (right)} & \text{if } a_{i,j} = a_{i-1,j} \\ ar \text{ (almost-right)} & \text{if } a_{i,j} = a_{i-1,j} + 1. \\ s \text{ (special)} & \text{otherwise} \end{cases}$$

We may then define two functions using these properties.

Definition 6. *The chain of an entry $a_{i,j}$ with $i > 1$ is*

$$(4.3) \quad c(a_{i,j}) = \begin{cases} 0 & \text{if } p_l(a_{i,j}) = l \text{ or } p_r(a_{i,j}) = r \\ (1-t)(1-q) & \text{otherwise} \end{cases}$$

and the slice of a property p is

$$(4.4) \quad g(p) = \begin{cases} -q & \text{if } p = l \\ t & \text{if } p = al \\ 1 & \text{if } p = r \\ -qt & \text{if } p = ar \\ 0 & \text{if } p = s \end{cases}$$

With these we define two more functions: the first is a generalization of the expressions $(-q)$ and $(1 - q)$ from Tokuyama's formula; and the second considers two entries of a GT pattern that are related to $a_{i,j}$, in particular $a_{i+1,j}$ and $a_{i+1,j+1}$.

Definition 7. For an entry $a_{i,j}$ with $i > 1$, we define

$$(4.5) \quad w(a_{i,j}) = c(a_{i,j}) + g(p_l(a_{i,j})) + g(p_r(a_{i,j})).$$

For an entry $a_{i,j}$ with $i < j < n$, we define

$$(4.6) \quad d(a_{i,j}) = g(p_r(a_{i+1,j})) \cdot g(p_l(a_{i+1,j+1})).$$

Example 4.1. Given the following segment of a GT pattern:

$$\begin{array}{ccc} 5 & 3 & 1 \\ 4 & 3 & \end{array}$$

We see that the 4 has properties al and ar . Thus $c(4) = (1 - q)(1 - t)$ and $w(4) = (1 - q)(1 - t) + t - qt = 1 - q$. Similarly, we have $c(3) = 0$ and $w(3) = 0 - q - 0 = -q$. For the entries in the second row, we find $g(p_r(4)) = -qt$ and $g(p_l(3)) = -q$, thus the 3 in the first row gives $d(3) = (-qt) \cdot (-q) = q^2t$.

Table 1 presents all possible values for $w(a_{i,j})$ and $d(a_{i,j})$ that we may need to consider.

TABLE 1. List of possible $w(a_{i,j})$ and $d(a_{i,j})$ values for an entry $a_{i,j}$.

$\{p_l(a_{i,j}), p_r(a_{i,j})\}$	$w(a_{i,j})$	$p_r(a_{i+1,j})$	$p_l(a_{i+1,j+1})$	$d(a_{i,j})$
(l, s)	$-q$	s	s	0
(s, r)	1	s	l, al	0
(l, ar)	$-q - qt$	r, ar	s	0
(al, r)	$1 + t$	r	l	$-q$
(s, ar)	$1 - q - t$	ar	l	q^2t
(al, s)	$1 - q + qt$	r	al	t
(al, ar)	$(1 - q)$	ar	al	$-qt^2$
(s, s)	$(1 - q)(1 - t)$			

Example 4.1 illustrates that to define $w(a_{i,j})$ and $d(a_{i,j})$, we only need to know two consecutive rows of a GT pattern. This leads us to the following definition.

Definition 8. Suppose λ is a strictly decreasing partition. We define $G_2(\lambda)$ to be the set of all partitions μ such that the length of μ is one less than that of λ , and $\lambda_i \geq \mu_i \geq \lambda_{i+1}$ for all i . For λ of length 1, we let $G_2(\lambda) = \{\emptyset\}$.

This definition ensures that μ would be a valid weakly decreasing row directly below a row λ in a GT pattern. Arranging λ and μ in this manner, we are able to extend Definition

7 to parts in λ and μ , and thus define $w(\mu_i)$ for all i and $d(\lambda_i)$ for $1 < i < n$. We require these in a special matrix.

Definition 9. Let λ and μ be partitions with $\mu \in G_2(\lambda)$. Then we define the square matrix $M(\lambda; \mu)$ as follows:

$$(4.7) \quad M(\lambda; \mu) = \begin{pmatrix} w(\mu_1) & 1 & 0 & \dots & 0 & 0 \\ d(\lambda_2) & w(\mu_2) & 1 & \dots & 0 & 0 \\ 0 & d(\lambda_3) & w(\mu_3) & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & w(\mu_{n-2}) & 1 \\ 0 & 0 & 0 & \dots & d(\lambda_{n-1}) & w(\mu_{n-1}) \end{pmatrix}$$

The determinant of $M(\lambda; \mu)$ is denoted $|M(\lambda; \mu)|$. If λ is of length 1, and $\mu = \emptyset \in G_2(\lambda)$, we assign $|M(\lambda; \mu)| = 1$. We also extend this notation to any pair $(\lambda; \mu)$ for all $\mu \notin G_2(\lambda)$ by defining $M(\lambda; \mu)$ to be the $(n-1)$ by $(n-1)$ zero matrix when λ has length n .

Thus, with these definitions, we are able to state a theorem which generalizes Theorem 2.2 due to Tokuyama to the Hall-Littlewood polynomials.

Theorem 4.1. For a partition λ of length n , we have

$$(4.8) \quad \Delta_n(q) \cdot R_\lambda(x; t) = \sum_{T \in SG(\lambda + \rho)} \prod_{i=1}^{n-1} \left(\sum_{\phi \in \Omega(r_{i+1})} t^{l(\phi)} |M(r_i; \phi(r_{i+1}))| \right) x^{m(T)}.$$

Note: We include an example of a computation of Theorem 4.1 for a particular GT pattern in Appendix A. Readers may prefer to visit this before continuing to view the theorem's proof in the next section.

5. PROOF

In this section, we prove Theorem 4.1. We begin by providing the reader with an outline of the proof:

GT patterns are inductive in nature because removing a top row reveals a smaller nested GT pattern. This yields a recursive statement of Theorem 4.1 which we present and prove as Theorem 5.1. To prove this, we further notice that cofactor expansion upon the matrix in Definition 9 yields another recursive aspect of the Hall-Littlewood polynomials, which links the Hall-Littlewood polynomials corresponding to the partition λ to those of the same partition with the first one or two parts removed. This result is given algebraically in Lemma 5.1 and Corollary 5.1 to follow. We use these ideas to show our main result.

Theorem 4.1 will be proven as a corollary to our next theorem, which is expressed as a summation over the set $G_2(\lambda + \rho)$. This theorem is based on the idea that GT patterns contain smaller nested GT patterns, which all have top rows that are elements of $G_2(\lambda + \rho)$. Prior to its statement we define some operators.

Let ζ_i be the permutation $(i \ i+1 \dots n \ n+1)$ in cycle notation which acts on $f(x)$ by

$$(5.1) \quad \zeta_i(f(x)) = f(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1}).$$

In the case of ζ_1 , we drop the subscript and write ζ . Similarly the permutation $\zeta_{i,j} = \zeta_{j,i}$ for $i < j$ is the composite permutation $(j+1 \ j+2 \ \dots \ n+1 \ n+2) \cdot (i \ i+1 \ \dots \ n \ n+1)$, which acts on $f(x)$ by

$$(5.2) \quad \zeta_{i,j}(f(x)) = \zeta_{j,i}(f(x)) = f(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{j-1}, x_{j+1}, \dots, x_{n+2}).$$

Furthermore, given a partition $\lambda = (\lambda_1, \dots, \lambda_n)$, we define

$$(5.3) \quad \lambda_{\hat{1}} = (\lambda_2, \dots, \lambda_n) \text{ and } \lambda_{\hat{1}, \hat{2}} = (\lambda_3, \dots, \lambda_n).$$

Theorem 5.1. For a partition λ of length n , we have

$$(5.4) \quad \Delta_n(q) \cdot R_\lambda(x; t) = \sum_{\mu \in G_2(\lambda + \rho)} |M(\lambda + \rho; \mu)| x_1^{|\lambda + \rho| - |\mu|} \zeta(\Delta_{n-1}(q) \cdot R_{\mu - \rho}(x; t)).$$

We will prove Theorem 5.1 by induction. It is easy to check that the base cases of $n = 1$ and $n = 2$ hold by considering all the possible sets of $G_2(\lambda + \rho)$ corresponding to λ with length 1 or 2 and their associated coefficients. We present the inductive hypotheses and several relevant lemmas before proving the inductive step to conclude our result.

Our inductive hypotheses for $n > 2$ are

$$(5.5) \quad R_{\lambda_{\hat{1}}}(x; t) \cdot \prod_{i=2}^{n-1} (x_1 - qx_i) = \sum_{\mu \in G_2(\lambda_{\hat{1}} + \rho)} |M(\lambda_{\hat{1}} + \rho; \mu)| x_1^{|\lambda_{\hat{1}} + \rho| - |\mu|} \zeta(R_{\mu - \rho}(x; t)),$$

and

$$(5.6) \quad R_{\lambda_{\hat{1}, \hat{2}}}(x; t) \cdot \prod_{i=2}^{n-2} (x_1 - qx_i) = \sum_{\mu \in G_2(\lambda_{\hat{1}, \hat{2}} + \rho)} |M(\lambda_{\hat{1}, \hat{2}} + \rho; \mu)| x_1^{|\lambda_{\hat{1}, \hat{2}} + \rho| - |\mu|} \zeta(R_{\mu - \rho}(x; t)).$$

Assuming these, we prove the formula in Theorem 5.1 for $n > 2$.

Lemma 5.1. $R_\lambda(x; t)$ can be expressed in terms of λ_1 and $R_{\lambda_{\hat{1}}}(x; t)$ as

$$(5.7) \quad R_\lambda(x; t) = \sum_{1 \leq i \leq n} x_i^{\lambda_1} \left(\prod_{\substack{1 \leq a \leq n \\ a \neq i}} \frac{x_i - tx_a}{x_i - x_a} \right) \zeta_i(R_{\lambda_{\hat{1}}}(x; t)).$$

Proof. Let the permutation $\psi_i = (i \ i-1 \ \dots \ 1)$, and let H be the symmetric group acting on the $(n-1)$ indices $(2, 3, \dots, n)$. Then by definition

$$R_\lambda(x; t) = \sum_{1 \leq i \leq n} \sum_{\sigma \in H} \psi_i \sigma \left(x_1^{\lambda_1} \prod_{2 \leq a \leq n} \frac{x_1 - tx_a}{x_1 - x_a} \zeta \left(\frac{x^{\lambda_{\hat{1}}} \Delta_{n-1}(t)}{\Delta_{n-1}} \right) \right).$$

Because $\sigma \in H$ does not permute x_1 , we have

$$R_\lambda(x; t) = \sum_{1 \leq i \leq n} \psi_i \left(x_1^{\lambda_1} \prod_{2 \leq a \leq n} \frac{x_1 - tx_a}{x_1 - x_a} \zeta(R_{\lambda_{\hat{1}}}(x; t)) \right),$$

which rearranges to the desired result. □

Corollary 5.1. $R_\lambda(x; t)$ can be expressed in terms of λ_1 , λ_2 and $R_{\lambda_{\hat{1}, \hat{2}}}(x; t)$ as

$$(5.8) \quad R_\lambda(x; t) = \sum_{1 \leq i \leq n} \sum_{\substack{1 \leq j \leq n \\ j \neq i}} x_i^{\lambda_1} x_j^{\lambda_2} \left(\prod_{\substack{1 \leq a \leq n \\ a \neq i}} \frac{x_i - tx_a}{x_i - x_a} \prod_{\substack{1 \leq a \leq n \\ a \neq i, j}} \frac{x_j - tx_a}{x_j - x_a} \right) \zeta_{i, j} \left(R_{\lambda_{\hat{1}, \hat{2}}}(x; t) \right).$$

Proof. Taking an equivalent expression of Lemma 5.1 in which we express $R_{\lambda_{\hat{1}}}(x; t)$ in terms of λ_2 and $R_{\lambda_{\hat{1}, \hat{2}}}(x; t)$, we replace this into the original expression of Lemma 5.1. Rearranging gives us the desired result. \square

We introduce a function that will be used in several of the upcoming lemmas. It is

$$(5.9) \quad F_\lambda(\kappa) = \sum_{\mu \in G_2(\lambda + \rho)} |M(\lambda + \rho; \mu)| x_1^{|\lambda + \rho| - |\mu|} \zeta \left(R_{\kappa||\mu - \rho}(x; t) \right).$$

Assuming the inductive hypotheses given in (5.5) and (5.6), we will use this notation in the following two lemmas to show a primary correspondence between the Hall-Littlewood polynomials and the set G_2 of various partitions.

Lemma 5.2. Suppose u is some nonnegative integer. Then, assuming the inductive hypothesis in (5.5), we have

$$(5.10) \quad F_{\lambda_{\hat{1}}}(u) = \sum_{2 \leq i \leq n} x_i^u \left(\prod_{\substack{2 \leq a \leq n \\ a \neq i}} \frac{x_i - tx_a}{x_i - x_a} \right) \zeta_i \left(R_{\lambda_{\hat{1}}}(x; t) \right) \prod_{\substack{2 \leq a \leq n \\ a \neq i}} (x_1 - qx_a).$$

Proof. By applying Lemma 5.1 to write $R_{u||(\mu - \rho)}(x; t)$ in terms of u and $R_{\mu - \rho}(x; t)$, the left hand side of the above equality becomes

$$\sum_{\mu \in G_2(\lambda_{\hat{1}} + \rho)} |M(\lambda_{\hat{1}} + \rho; \mu)| x_1^{|\lambda_{\hat{1}} + \rho| - |\mu|} \zeta \left(\sum_{1 \leq i \leq n-1} x_i^u \left(\prod_{\substack{1 \leq a \leq n-1 \\ a \neq i}} \frac{x_i - tx_a}{x_i - x_a} \right) \zeta_i \left(R_{\mu - \rho}(x; t) \right) \right).$$

We rearrange this expression to write this as

$$\sum_{2 \leq i \leq n} x_i^u \left(\prod_{\substack{2 \leq a \leq n \\ a \neq i}} \frac{x_i - tx_a}{x_i - x_a} \right) \zeta_i \left(\sum_{\mu \in G_2(\lambda_{\hat{1}} + \rho)} |M(\lambda_{\hat{1}} + \rho; \mu)| x_1^{|\lambda_{\hat{1}} + \rho| - |\mu|} \zeta \left(R_{\mu - \rho}(x; t) \right) \right).$$

Finally, we replace the expression to the right of ζ_i using the inductive hypothesis in (5.5) to give the desired result. \square

Lemma 5.3. Suppose u and v are some nonnegative integers. Then, assuming the inductive hypothesis in (5.6), we have

$$(5.11) \quad F_{\lambda_{\hat{1}, \hat{2}}}(u, v) = \sum_{2 \leq i \leq n} \sum_{\substack{2 \leq j \leq n \\ j \neq i}} x_i^u x_j^v h_{i, j}(\lambda_{\hat{1}, \hat{2}}) \prod_{\substack{2 \leq a \leq n \\ a \neq i, j}} (x_1 - qx_a),$$

where

$$(5.12) \quad h_{i,j}(\lambda) = \left(\prod_{\substack{2 \leq a \leq n \\ a \neq i}} \frac{x_i - tx_a}{x_i - x_a} \prod_{\substack{2 \leq a \leq n \\ a \neq i,j}} \frac{x_j - tx_a}{x_j - x_a} \right) \zeta_{i,j}(R_\lambda(x; t)).$$

Proof. The proof of this lemma is very similar to that of the previous Lemma 5.2. For brevity we only present the outline of the proof.

Recalling the definition of $F_\lambda(\kappa)$, we rewrite the left hand side using Corollary 5.1 to express $R_{((u,v)||(\mu-\rho))}(x; t)$ in terms of u , v , and $R_{\mu-\rho}(x; t)$. Then, upon rearranging (in a similar fashion to Lemma 5.2), we directly apply the inductive hypothesis in 5.6 to obtain the desired result. \square

For the next two lemmas, we recognize that Theorem 5.1 uses the determinant of a matrix. Cofactor expansion around its first column yields the recursion

$$(5.13) \quad |M(\lambda + \rho; \mu)| = w(\mu_1)|M(\lambda_{\widehat{1}} + \rho; \mu_{\widehat{1}})| - d(\lambda_2)|M(\lambda_{\widehat{1}, \widehat{2}} + \rho; \mu_{\widehat{1}, \widehat{2}})|.$$

The next two lemmas address each of the two determinants on the right hand side respectively.

Lemma 5.4. Let $O_i = ((x_i - tx_1)x_i^{\lambda_1} - (x_1 - tx_i)x_1^{\lambda_1 - \lambda_2}x_i^{\lambda_2})/(x_i - tx_1)$. Then we have

$$(5.14) \quad \sum_{\mu \in G_2(\lambda + \rho)} w(\mu_1)|M(\lambda_{\widehat{1}} + \rho; \mu_{\widehat{1}})|x_1^{|\lambda + \rho| - |\mu|} \zeta(R_{\mu - \rho}(x; t)) \\ = \sum_{2 \leq i \leq n} O_i \left(\prod_{\substack{1 \leq a \leq n \\ a \neq i}} \frac{x_i - tx_a}{x_i - x_a} \right) \zeta_i(R_{\lambda_{\widehat{1}}}(x; t)) \prod_{i=2}^n (x_1 - qx_i).$$

Proof. First, we claim: $(x_i - tx_1)O_i/(x_i - x_1) = Q_i/(x_1 - qx_i)$,

$$Q_i = -qx_i^{\lambda_1 + 1} + tx_1x_i^{\lambda_1} - qtx_1^{\lambda_1 - \lambda_2}x_i^{\lambda_2 + 1} \\ + x_1^{\lambda_1 - \lambda_2 + 1}x_i^{\lambda_2} + \sum_{\lambda_2 < i \leq \lambda_1} (1 - q)(1 - t)x_1^{\lambda_1 + 1 - i}x_i^i.$$

This can be shown through simple algebraic manipulation, considering three cases for λ , namely (1): $\lambda_1 = \lambda_2$, (2): $\lambda_1 = 1 + \lambda_2$ and (3): $\lambda_1 > 1 + \lambda_2$.

Substituting the claim in the right hand side of the lemma gives

$$\text{RHS} = \sum_{2 \leq i \leq n} \left(\prod_{\substack{2 \leq a \leq n \\ a \neq i}} \frac{x_i - tx_a}{x_i - x_a} \right) \zeta_i(R_{\lambda_{\widehat{1}}}(x; t)) Q_i \prod_{\substack{2 \leq a \leq n \\ a \neq i}} (x_1 - qx_a).$$

Then, expanding Q_i and applying Lemma 5.2, we have

$$\text{RHS} = -q \cdot F_{\lambda_{\widehat{1}}}(\lambda_1 + 1) + tx_1 \cdot F_{\lambda_{\widehat{1}}}(\lambda_1) - qtx_1^{\lambda_1 - \lambda_2} \cdot F_{\lambda_{\widehat{1}}}(\lambda_2 + 1) \\ + x_1^{\lambda_1 - \lambda_2 + 1} \cdot F_{\lambda_{\widehat{1}}}(\lambda_2) + \sum_{\lambda_2 < i \leq \lambda_1} (1 - q)(1 - t)x_1^{\lambda_1 + 1 - i} \cdot F_{\lambda_{\widehat{1}}}(i).$$

Examining Definition 6, we see that the first two coefficients above are precisely the nonzero possibilities of $g(p_l(\mu_1))$; the next two are precisely the nonzero possibilities of $g(p_r(\mu_1))$; and the final summation is over all the nonzero possibilities of $c(\mu_1)$. Recalling from Definition 7 that $w(\mu_1) = c(\mu_1) + g(p_l(\mu_1)) + g(p_r(\mu_1))$, we simply have

$$(5.15) \quad \text{RHS} = \sum_{\mu \in G_2(\lambda + \rho)} w(\mu_1) |M(\lambda_{\widehat{1}} + \rho; \mu_{\widehat{1}})| x_1^{|\lambda + \rho| - |\mu|} \zeta(R_{\mu - \rho}(x; t)). \quad \square$$

Lemma 5.5. We have

$$(5.16) \quad \sum_{\mu \in G_2(\lambda + \rho)} d(\lambda_2) |M(\lambda_{\widehat{1}, \widehat{2}} + \rho; \mu_{\widehat{1}, \widehat{2}})| x_1^{|\lambda + \rho| - |\mu|} \zeta(R_{\mu - \rho}(x; t)) \\ = x_1^{\lambda_1 - \lambda_2} \sum_{2 \leq i \leq n} \sum_{\substack{2 \leq j \leq n \\ j \neq i}} t x_i^{\lambda_2} x_j^{\lambda_2} h_{i,j}(\lambda_{\widehat{1}, \widehat{2}}) \prod_{a=2}^n (x_1 - q x_a),$$

where $h_{i,j}(\lambda)$ was defined in Lemma 5.3.

Proof. Expanding the factor $(x_1 - q x_i)(x_1 - q x_j)$ from the product $\prod_{a=2}^n (x_1 - q x_a)$, the right hand side of the above equality becomes

$$\sum_{2 \leq i \leq n} \sum_{\substack{2 \leq j \leq n \\ j \neq i}} t x_1^{\lambda_1 - \lambda_2} x_i^{\lambda_2} x_j^{\lambda_2} (q^2 x_i x_j - q x_1 x_i - q x_1 x_j + x_1^2) h_{i,j}(\lambda_{\widehat{1}, \widehat{2}}) \prod_{\substack{2 \leq a \leq n \\ a \neq i, j}} (x_1 - q x_a).$$

We see from Proposition 3.1 that $F_{\lambda_{\widehat{1}, \widehat{2}}}(\lambda_2, \lambda_2 + 1) = t \cdot F_{\lambda_{\widehat{1}, \widehat{2}}}(\lambda_2 + 1, \lambda_2)$. Then distributing $(q^2 x_i x_j - q x_1 x_i - q x_1 x_j + x_1^2)$ over the summation and applying Lemma 5.3 yields

$$\begin{aligned} \text{RHS} &= q^2 t x_1^{\lambda_1 - \lambda_2} \cdot F_{\lambda_{\widehat{1}, \widehat{2}}}(\lambda_2 + 1, \lambda_2 + 1) - q x_1^{\lambda_1 - \lambda_2 + 1} \cdot F_{\lambda_{\widehat{1}, \widehat{2}}}(\lambda_2, \lambda_2 + 1) \\ &\quad - q t^2 x_1^{\lambda_1 - \lambda_2 + 1} \cdot F_{\lambda_{\widehat{1}, \widehat{2}}}(\lambda_2 + 1, \lambda_2) + t x_1^{\lambda_1 - \lambda_2 + 2} \cdot F_{\lambda_{\widehat{1}, \widehat{2}}}(\lambda_2, \lambda_2, \mu - \rho). \end{aligned}$$

Recalling from Definition 6 that $d(\lambda_2) = g(p_r(\mu_1)) \cdot g(p_l(\mu_2))$, we notice that each of the four coefficients in the previous expression corresponds exactly to each of the four possible nonzero values for $d(\lambda_2)$. Thus, we have

$$\text{RHS} = \sum_{\mu \in G_2(\lambda + \rho)} d(\lambda_2) |M(\lambda_{\widehat{1}, \widehat{2}} + \rho; \mu_{\widehat{1}, \widehat{2}})| x_1^{|\lambda + \rho| - |\mu|} \zeta(R_{\mu - \rho}(x; t)). \quad \square$$

With these lemmas in mind, we return to the proof of Theorem 5.1.

Proof. We begin with the following claim:

$$0 = \sum_{2 \leq i \leq n} \sum_{2 \leq j \leq n} (-1)^{i+j} x_i^{\lambda_2} x_j^{\lambda_2} (x_i - x_j) \zeta_{i,j}(\Delta_{n-2} \cdot R_{\lambda_{\widehat{1}, \widehat{2}}}(x; t)) \prod_{\substack{2 \leq a \leq n \\ a \neq i}} (x_i - t x_a) \prod_{\substack{2 \leq a \leq n \\ a \neq j}} (x_j - t x_a).$$

This can easily be seen by swapping the subscripts i and j in the right hand side, revealing $\text{RHS} = -\text{RHS}$.

We divide through the equality above by Δ_n , altering the products and the bounds of the summation, and multiply by $x_1(1-t)$ to find

$$0 = \sum_{2 \leq i \leq n} \sum_{\substack{2 \leq j \leq n \\ j \neq i}} \frac{x_1(1-t)(x_j - tx_i)x_i^{\lambda_2}x_j^{\lambda_2}}{(x_j - x_1)(x_i - tx_1)} \zeta_{i,j}(R_{\lambda_{\widehat{1},\widehat{2}}}(x;t)) \prod_{\substack{1 \leq a \leq n \\ a \neq i}} \frac{x_i - tx_a}{x_i - x_a} \prod_{\substack{2 \leq a \leq n \\ a \neq i,j}} \frac{x_j - tx_a}{x_j - x_a}.$$

Using the identity

$$\frac{x_1(1-t)(x_j - tx_i)}{(x_j - x_1)(x_i - tx_1)} = \frac{(x_1 - tx_i)(x_j - tx_1)}{(x_i - tx_1)(x_j - x_1)} + t \frac{x_i - x_1}{x_i - tx_1},$$

we break the double summation into two parts; in particular, if we take

$$(5.17) \quad L = \sum_{2 \leq i \leq n} \sum_{\substack{2 \leq j \leq n \\ j \neq i}} tx_i^{\lambda_2}x_j^{\lambda_2}h_{i,j}(\lambda_{\widehat{1},\widehat{2}}),$$

where $h_{i,j}(\lambda)$ is defined in Lemma 5.3, then

$$(5.18) \quad 0 = L + \sum_{2 \leq i \leq n} \sum_{\substack{2 \leq j \leq n \\ j \neq i}} \frac{(x_1 - tx_i)x_i^{\lambda_2}x_j^{\lambda_2}}{(x_i - tx_1)} \zeta_{i,j}(R_{\lambda_{\widehat{1},\widehat{2}}}(x;t)) \prod_{\substack{1 \leq a \leq n \\ a \neq i}} \frac{x_i - tx_a}{x_i - x_a} \prod_{\substack{1 \leq a \leq n \\ a \neq i,j}} \frac{x_j - tx_a}{x_j - x_a}.$$

Consider the claim:

$$(5.19) \quad -x_1^{\lambda_2} \left(\prod_{2 \leq a \leq n} \frac{x_1 - tx_a}{x_1 - x_a} \right) \zeta(R_{\lambda_{\widehat{1}}}(x;t)) \\ = x_1^{\lambda_2} \sum_{2 \leq i \leq n} \left(\prod_{\substack{1 \leq a \leq n \\ a \neq i}} \frac{x_i - tx_a}{x_i - x_a} \prod_{\substack{2 \leq a \leq n \\ a \neq i}} \frac{x_1 - tx_a}{x_1 - x_a} \right) \frac{(x_1 - tx_i)x_i^{\lambda_2}}{(x_i - tx_1)} \zeta_{1,i}(R_{\lambda_{\widehat{1},\widehat{2}}}(x;t)).$$

This holds by expressing $R_{\lambda_{\widehat{1}}}(x;t)$ of the left hand side explicitly using Lemma 5.1 and rearranging the consequent result.

We notice that the right hand side of (5.19) is equivalent to setting $j = 1$ in the double summation of (5.18). Thus, adding either side of (5.19) to either side of (5.18) respectively, we have

$$-x_1^{\lambda_2} \left(\prod_{2 \leq a \leq n} \frac{x_1 - tx_a}{x_1 - x_a} \right) \zeta(R_{\lambda_{\widehat{1}}}(x;t)) \\ = L + \sum_{2 \leq i \leq n} \sum_{\substack{1 \leq j \leq n \\ j \neq i}} \frac{(x_1 - tx_i)x_i^{\lambda_2}x_j^{\lambda_2}}{(x_i - tx_1)} \zeta_{i,j}(R_{\lambda_{\widehat{1},\widehat{2}}}(x;t)) \prod_{\substack{1 \leq a \leq n \\ a \neq i}} \frac{x_i - tx_a}{x_i - x_a} \prod_{\substack{1 \leq a \leq n \\ a \neq i,j}} \frac{x_j - tx_a}{x_j - x_a}.$$

Multiplying through by $-x_1^{\lambda_1 - \lambda_2}$, and adding

$$\sum_{2 \leq i \leq n} x_i^{\lambda_1} \left(\prod_{\substack{1 \leq a \leq n \\ a \neq i}} \frac{x_i - tx_a}{x_i - x_a} \right) \zeta_i(R_{\lambda_{\widehat{1}}}(x;t)),$$

to both sides, we may apply Lemma 5.1 once on either side of the equality to obtain

$$R_\lambda(x; t) = -x_1^{\lambda_1 - \lambda_2} L + \sum_{2 \leq i \leq n} \left(\prod_{\substack{1 \leq a \leq n \\ a \neq i}} \frac{x_i - tx_a}{x_i - x_a} \right) \left(x_i^{\lambda_1} - x_1^{\lambda_1 - \lambda_2} x_i^{\lambda_2} \frac{x_1 - tx_i}{(x_i - tx_1)} \right) \zeta_i(R_{\lambda_{\hat{1}}}(x; t)).$$

Recalling O_i from Lemma 5.4 as

$$O_i = \frac{(x_i - tx_1)x_i^{\lambda_1} - (x_1 - tx_i)x_1^{\lambda_1 - \lambda_2} x_i^{\lambda_2}}{x_i - tx_1} = x_i^{\lambda_1} - x_1^{\lambda_1 - \lambda_2} x_i^{\lambda_2} \frac{x_1 - tx_i}{x_i - tx_1},$$

we combine the two summations on the right hand side and recall L from (5.17) to write

$$R_\lambda(x; t) = \sum_{2 \leq i \leq n} O_i \left(\prod_{\substack{1 \leq a \leq n \\ a \neq i}} \frac{x_i - tx_a}{x_i - x_a} \right) \zeta_i(R_{\lambda_{\hat{1}}}(x; t)) - x_1^{\lambda_1 - \lambda_2} \sum_{2 \leq i \leq n} \sum_{\substack{2 \leq j \leq n \\ j \neq i}} tx_i^{\lambda_2} x_j^{\lambda_2} h_{i,j}(\lambda_{\hat{1}, \hat{2}}).$$

Multiplying throughout by $\prod_{i=2}^n (x_1 - qx_i)$, we apply Lemmas 5.4 and 5.5 to the respective summations on the right hand side to find

$$\begin{aligned} \prod_{i=2}^n (x_1 - qx_i) \cdot R_\lambda(x; t) &= \sum_{\mu \in G_2(\lambda + \rho)} w(\mu_1) |M(\lambda_{\hat{1}} + \rho; \mu_{\hat{1}})| x_1^{|\lambda + \rho| - |\mu|} \zeta(R_{\mu - \rho}(x; t)) \\ &\quad - \sum_{\mu \in G_2(\lambda + \rho)} d(\lambda_2) |M(\lambda_{\hat{1}, \hat{2}} + \rho; \mu_{\hat{1}, \hat{2}})| x_1^{|\lambda + \rho| - |\mu|} \zeta(R_{\mu - \rho}(x; t)). \end{aligned}$$

Finally, recalling from (5.13) that $|M(\lambda + \rho; \mu)| = w(\mu_1) |M(\lambda_{\hat{1}} + \rho; \mu_{\hat{1}})| - d(\lambda_2) |M(\lambda_{\hat{1}, \hat{2}} + \rho; \mu_{\hat{1}, \hat{2}})|$, we multiply by $\zeta(\Delta_{n-1})$ to conclude

$$\Delta_n(q) \cdot R_\lambda(x; t) = \sum_{\mu \in G_2(\lambda + \rho)} |M(\lambda + \rho; \mu)| x_1^{|\lambda + \rho| - |\mu|} \zeta(\Delta_{n-1}(q) \cdot R_{\mu - \rho}(x; t)). \quad \square$$

Theorem 4.1. For a partition λ of length n , we have

$$(4.8) \quad \Delta_n(q) \cdot R_\lambda(x; t) = \sum_{T \in SG(\lambda + \rho)} \prod_{i=1}^{n-1} \left(\sum_{\phi \in \Omega(r_{i+1})} t^{l(\phi)} |M(r_i; \phi(r_{i+1}))| \right) x^{m(T)}.$$

Proof. We prove this using mathematical induction, where our inductive hypothesis is that Theorem 4.1 holds for all Hall-Littlewood polynomials in $(n-1)$ variables. The base case formula of one variable is easy to check.

We now prove the formula holds for $n > 1$ assuming the inductive hypothesis. We notice that all elements $\mu \in G_2(\lambda + \rho)$ in Theorem 5.1 have the property that $\lambda + \rho$ is strictly decreasing, whereas μ need not be. If μ is not strictly decreasing, the computed Hall-Littlewood polynomial $R_{\mu - \rho}(x; t)$ has an increasing partition $\mu - \rho$. Proposition 3.1 allows us to express these Hall-Littlewood polynomials in terms of some Hall-Littlewood polynomials that are obtained from strictly decreasing $\kappa \in G_2(\lambda + \rho)$. In particular we have that for all non-strict $\mu \in G_2(\lambda + \rho)$, there exists a unique strict $\kappa \in G_2(\lambda + \rho)$ and a unique element $\phi \in \Omega(\kappa)$ such that $\phi(\kappa) = \mu$. Furthermore any tuple $\phi(\kappa)$ obtained from $\phi \in \Omega(\kappa)$ will either be a valid element of $G_2(\lambda + \rho)$ or will cause $|M(\lambda + \rho; \phi(\kappa))| = 0$ and can be

neglected. Hence, we may rewrite Theorem 5.1 as the following summation over strictly decreasing $\mu \in G_2(\lambda + \rho)$:

$$(5.20) \quad R_\lambda(x; t) \cdot \Delta_n(q) = \sum_{\substack{\mu \in G_2(\lambda + \rho) \\ \mu \text{ strict}}} \sum_{\phi \in \Omega(\mu)} t^{l(\phi)} |M(\lambda + \rho; \phi(\mu))|_{x_1^{|\lambda + \rho| - |\mu|}} \zeta(\Delta_{n-1}(q) \cdot R_{\mu - \rho}(x; t)).$$

By applying the inductive hypothesis to the Hall-Littlewood polynomials $R_{\mu - \rho}(x; t)$ of $(n-1)$ variables in the identity above, we are reduced to the formula in Theorem 4.1. \square

6. COMPLEMENTS

Examining Theorem 4.1, we see that for each given GT pattern, we compute a product ranging from $i = 1$ to $i = n - 1$, which is essentially an iteration through each row. However, in Theorem 2.2, Tokuyama does not need this iteration and is able to consider each GT pattern as a whole. Thus, we provide an alternate statement of Theorem 4.1 that removes the need to consider each row individually.

Instead of considering the set Ω of each individual row, we extend its definition to an entire strict GT pattern T by taking the Cartesian product

$$(6.1) \quad \Omega(T) = \prod_{i=2}^n \Omega(r_i).$$

Let the $(n-1)$ -tuple ϕ_T be an element of $\Omega(T)$, such that $\phi_T = (\phi_2, \phi_3, \dots, \phi_n)$. Then ϕ_T acts on T canonically by having every ϕ_i in the tuple act on the row r_i .

We see that the action of ϕ_T on a GT pattern T reveals a new pattern, which we denote as $\phi_T(T)$. Figure 1 provides an example of a GT pattern and a possible corresponding pattern $\phi_T(T)$. It is worth noticing that the pattern $R_T(T)$ is not necessarily a GT pattern as it need not satisfy the condition $a_{i-1, j-1} \geq a_{i, j} \geq a_{i-1, j}$.



FIGURE 1. A GT pattern T and a corresponding pattern $\phi_T(T)$ with $\phi_T = \{[1 \ 2], \text{Id}\}$.

The matrix $M(r_i; \phi(r_{i+1}))$ in Theorem 4.1 implicitly considers the relationship between entries in row r_{i+1} after the action of the raising operator ϕ , and entries in the row r_i in its original form. When we extend the definition of the set Ω to act on the entire pattern T , therefore acting on both rows r_i and r_{i+1} at once, we modify the definitions of the *left-sided property*, *right-sided property*, *chain* and *slice* to account for this change.

Definition 10. For an entry $a_{i, j}$ in T , let $a'_{i, j}$ denote the corresponding entry in $\phi_T(T)$ after the action of the raising operator ϕ_T on T .

Now, we reassign the left-sided properties of an entry $p_l(a'_{i,j})$ as follows:

$$(6.2) \quad p_l(a'_{i,j}) = \begin{cases} l \text{ (left)} & \text{if } a'_{i,j} = a_{i-1,j-1} \\ al \text{ (almost-left)} & \text{if } a'_{i,j} = a_{i-1,j-1} - 1 \\ o \text{ (outward)} & \text{if } a'_{i,j} > a_{i-1,j-1} \\ s \text{ (special)} & \text{otherwise} \end{cases}$$

Similarly, we reassign the right-sided properties of an entry $p_r(a'_{i,j})$ as follows:

$$(6.3) \quad p_r(a'_{i,j}) = \begin{cases} r \text{ (right)} & \text{if } a'_{i,j} = a_{i-1,j} \\ ar \text{ (almost-right)} & \text{if } a'_{i,j} = a_{i-1,j} + 1 \\ o \text{ (outward)} & \text{if } a'_{i,j} < a_{i-1,j} \\ s \text{ (special)} & \text{otherwise} \end{cases}$$

Note that these are the same definitions given in (4.1) and (4.2), with the only difference being that we compare the entry $a'_{i,j}$ of a pattern *after* it has been acted on by the raising operator ϕ_T , to the appropriate entry in the row r_{i-1} of the *original* GT pattern.

In a similar manner to the above, we take the *chain* to be a function of $a'_{i,j}$, with the only alteration from Definition 6 being that the chain of any entry with an *outward* property is zero i.e. $c(o) = 0$. We also extend *slice* such that $g(o) = 0$.

We observe that the *outward* label only serves to implicitly disallow any $\phi_T(T)$ patterns which contain a row $\phi_i(r_i) \notin G_2(r_{i-1})$ by reducing its coefficient to zero. This is analogous to defining $M(\lambda; \mu)$ to be the $(n-1)$ by $(n-1)$ zero matrix for any $\mu \notin G_2(\lambda)$.

Let S denote a tuple of entries $a'_{i,j}$ with $i > 1$ in some pattern $\phi_T(T)$. The ordering of S is specified as follows: if a'_{i_1,j_1} precedes a'_{i_2,j_2} in S , then either $i_1 < i_2$, or both $i_1 = i_2$ and $j_1 < j_2$. We also have the complementary tuple of S , denoted as \widehat{S} , that contains all entries $a'_{i,j}$ with $i > 1$ that are not in S . Then, given a GT pattern T , we define $S(T)$ to be the set of all tuples S of T , and we similarly define $S(\phi_T(T))$.

Example 6.1. Given the following GT pattern:

$$\begin{array}{ccc} 6 & 3 & 0 \\ 5 & 1 & \\ & 2 & \end{array}$$

We have $S(T) = \{\emptyset, (5), (1), (2), (5, 1), (5, 2), (1, 2), (5, 1, 2)\}$.

Let $Z(S)$ be the set of all tuples obtained by the following Cartesian product maintaining the order of S :

$$(6.4) \quad Z(S) = \prod_{a'_{i,j} \in S} \{p_l(a'_{i,j}), p_r(a'_{i,j})\}.$$

Given a tuple $z \in Z(S)$, we define $\chi(z)$ as

$$(6.5) \quad \chi(z) = \begin{cases} 0 & \text{if there exists } p_r(a'_{i,j}) \text{ and } p_l(a'_{i,j+1}) \text{ in } z \\ 1 & \text{otherwise} \end{cases}.$$

In this manner, every element of $Z(S)$ can be viewed as a choice of either the left-sided or the right-sided property of each entry in S , with $\chi(z)$ acting as a restriction on the possible choices. With these in hand, we have the following definition:

Definition 11. For $z \in Z(S)$ we define $G(S)$ as

$$(6.6) \quad G(S) = \sum_{z \in Z(S)} \chi(z) \prod_{p \in z} g(p),$$

where $G(S) = 1$ when $S = \emptyset$.

Example 6.2. Using the GT pattern in Example 6.1, take $S = (5, 1)$. We then see that $\{p_l(5), p_r(5)\} = \{al, s\}$ and $\{p_l(1), p_r(1)\} = \{s, ar\}$. Thus, if we call

$$z_1 = (al, s) \quad z_2 = (al, ar) \quad z_3 = (s, s) \quad z_4 = (s, ar),$$

then we have $Z(S) = \{z_1, z_2, z_3, z_4\}$, with $\chi(z_3) = 0$ and $\chi(z_1) = \chi(z_2) = \chi(z_4) = 1$. Furthermore, we obtain $\prod_{p \in z} g(p) = 0$ for $z = z_1, z_3, z_4$ and $\prod_{p \in z} g(p) = -qt^2$ for $z = z_2$, giving $G(S) = -qt^2$.

Additionally we have the following definition:

Definition 12. Given a tuple of entries S from a GT pattern, we have

$$(6.7) \quad C(S) = \prod_{a'_{i,j} \in S} c(a'_{i,j})$$

where $C(S) = 1$ when $S = \emptyset$.

With these definitions in hand, we are able to state a complement to Theorem 4.1.

Theorem 6.1. For a partition λ of length n , we have

$$(6.8) \quad \Delta_n(q) \cdot R_\lambda(x; t) = \sum_{T \in SG(\lambda + \rho)} \left(\sum_{\phi_T \in \Omega(T)} t^{l(\phi_T)} \sum_{S \in S(\phi_T(T))} C(S) G(\hat{S}) \right) x^{m(T)}$$

The new coefficient obtained from the summation of $C(S)G(\hat{S})$ can be viewed as an expansion of the determinants of $M(r_i; \phi(r_{i+1}))$ in Theorem 4.1. In this manner, Theorem 6.1 provides more insight into the structure of the matrix $M(r_i, \phi(r_{i+1}))$ by elucidating the choosing process that occurs during the computation of the determinant of M . This theorem also eliminates the need to consider each row separately by extending the set Ω to act on the entire GT pattern, as opposed to one row at a time. It thus offers an interesting perspective on Theorem 4.1.

7. SPECIALIZATIONS

The results of this paper generalize Tokuyama's formula and several other existing results. We demonstrate a few of these specializations.

Tokuyama's formula. We know that for all raising operators $\phi \neq \text{Id}$, the length of ϕ is at least 1. Therefore, examining Theorem 4.1, setting $t = 0$ reduces the equation to

$$\Delta_n(q) \cdot s_\lambda = \sum_{T \in SG(\lambda + \rho)} \prod_{i=1}^{n-1} |M(r_i; r_{i+1})| x^{m(T)}.$$

As these are all the identity cases on strict Gelfand-Tsetlin patterns, every row is strictly decreasing. This implies that consecutive entries cannot have $p_r(a_{i,j}) = r$ and $p_l(a_{i,j+1}) = l$, and consequently $d(a_{i,j})$ cannot be $-q$. All the remaining possibilities of $d(a_{i,j})$ are reduced to 0 when $t = 0$. Thus, if we let $r_{i+1} = (\mu_1, \dots, \mu_{n-i})$, the matrix $M(r_i; r_{i+1})$ simplifies to

$$M(r_i; r_{i+1}) = \begin{pmatrix} w(\mu_1) & 1 & \dots & 0 \\ 0 & w(\mu_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & w(\mu_{n-i}) \end{pmatrix}$$

Therefore, we have

$$(7.1) \quad |M(r_i; r_{i+1})| = \prod_{k=1}^{n-i} w(\mu_k).$$

Finally, returning to Tokuyama's terminology of left-leaning, right-leaning and special entries from Definition 3, we find that $w(a_{i,j})$ simplifies to

$$(7.2) \quad w(a_{i,j}) = \begin{cases} -q & \text{if } a_{i,j} \text{ is left-leaning} \\ 1 - q & \text{if } a_{i,j} \text{ is special} \\ 1 & \text{if } a_{i,j} \text{ is right-leaning} \end{cases},$$

and, substituting this into (7.1) from earlier, we conclude with Tokuyama's formula:

$$\Delta_n(q) \cdot s_\lambda = \sum_{T \in SG(\lambda + \rho)} (-q)^{l(T)} (1 - q)^{s(T)} x^{m(T)}.$$

Comparing the results of this paper with Tokuyama's formula reveals some interesting distinctions regarding the structure of the Hall-Littlewood polynomials in relation to the Schur polynomials. Theorem 5.1 demonstrates that when expressing $R_\lambda(x; t)$ recursively in terms of $R_{\mu-\rho}(x; t)$, it is more natural to include several non-strictly decreasing partitions μ in the summation. This is not an issue for Tokuyama's formula as Schur polynomials of such non-strictly decreasing partitions μ are just $s_{\mu-\rho}(x) = 0$.

In fact, whilst Theorems 4.1 and 6.1 are stated as summations over strict GT patterns, the use of Ω is to allow an implicit summation over all possible non-strict rows. We thus naturally seek a way to consider the contributions of such rows directly, eliminating the more *ad hoc* use of Ω . Both theorems also highlight the added complexity in a Hall-Littlewood polynomial as they account for the ordering among the entries in a GT pattern instead of simply counting entries as Tokuyama does with $s(T)$ and $l(T)$.

Stanley's formula. In [Sta86], Stanley gave a formula for the Hall-Littlewood polynomials at the singular value $t = -1$, also known as the Schur q -functions, as a generating function of strict GT patterns of top row λ :

$$(7.3) \quad R_\lambda(x; -1) = \sum_{T \in SG(\lambda)} 2^{s(T)} x^{m(T)}$$

Tokuyama subsequently showed in [Tok88] that his formula yields (7.3) when the deformation parameter q is set to -1 . Theorems 4.1 and 6.1 thus specialize to (7.3) at $t = 0$ and $q = -1$, by virtue of specializing to Tokuyama's result. However, setting t to -1 in Theorem 4.1 also

gives a deformation along q of (7.3), and we can show that this deformation reduces to (7.3) at $q = 0$.

Examining Theorem 6.1, we see that any pattern containing an entry with $p_l = l$ gives an overall coefficient of zero, so we may simply sum over the set $SGL(\lambda + \rho) \subset SG(\lambda + \rho)$ that contains all GT patterns without l entries. Also, since for every non-trivial $\phi \in \Omega(T)$ the pattern $\phi(T)$ contains either an l or an o entry, we need only consider $\phi = \text{Id}$. Furthermore, Theorem 4.1 reveals that any GT pattern containing consecutive r then al entries yields a coefficient $D = \prod_{i=2}^n |M(r_{i-1}; r_i)| = 0$ as follows:

Assume for the sake of contradiction that there is a GT pattern with consecutive r then al entries for which $D \neq 0$. Let $a_{i,j}$ and $a_{i,j+1}$ be the lowest consecutive r then al entries (there may be others on the same row, but none below). Then the r entry $a_{i,j} = u$ for some $u \in \mathbb{N}$, and the al entry $a_{i,j+1} = u - 1$. Now consider the entry $a_{i+1,j+1}$. Since it cannot be l , or else $D = 0$, we must have $a_{i+1,j+1} = u - 1$. Then $w(a_{i+1,j}) = 0$; so to ensure $M(r_i; r_{i+1}) \neq 0$ (and consequently that $D \neq 0$), we must have that either $p_r(a_{i+1,j}) = r$ or $p_l(a_{i+1,j+2}) = al$. But this contradicts our hypothesis. Therefore, any GT pattern containing consecutive r then al entries yields an overall coefficient of zero.

Let the set $SGL^*(\lambda + \rho) \subset SGL(\lambda + \rho)$ contain all GT patterns without l entries or consecutive r then al entries. Then Theorem 4.1 simplifies to

$$(7.4) \quad \Delta_n(0) \cdot R_\lambda(x; -1) = \sum_{T \in SGL^*(\lambda + \rho)} \prod w(a_{i,j}) x^{m(T)},$$

where the product is taken over all possible $a_{i,j}$. This leads us to introduce a bijective mapping, similar to one used in [Tok88]:

$$(7.5) \quad \begin{aligned} SGL^*(\lambda + \rho) &\longrightarrow SG(\lambda) \\ \Theta : \quad a_{i,j} &\longmapsto a_{i,j} + j - n \\ m(T) &\longmapsto m(T) - \rho \end{aligned}.$$

After applying this mapping, we have that $w(a_{i,j}) = 2$ for all *special* entries and $w(a_{i,j}) = 1$ otherwise, and thus (7.4) reduces to

$$x^\rho \cdot R_\lambda(x; -1) = x^\rho \cdot \sum_{T \in SG(\lambda)} 2^{s(T)} x^{m(T)},$$

Dividing out by x^ρ , we obtain (7.3).

Monomial symmetric function. Theorem 6.1 also simplifies significantly in the case of $t = 1$. Recalling Definition 6 of $c(a_{i,j})$, we see that setting $t = 1$ results in $c(a_{i,j}) = 0$ for any entry $a_{i,j}$. Thus, the function $C(S)$ from Definition 12 is only nonzero when $S = \emptyset$. Then, we have

$$\sum_{S \in S(\phi_T(T))} C(S) G(\hat{S}) = G(K),$$

where K is the tuple containing all entries of $\phi_T(T)$. Therefore, we can write

$$(7.6) \quad \Delta_n(q) \cdot m_\lambda = \sum_{T \in SG(\lambda + \rho)} x^{m(T)} \sum_{\phi_T \in \Omega(T)} G(K).$$

We also observe that $G(K)$ is a polynomial in q that can be written as some summation of the form $G(K) = \sum (-q)^k$. Thus specializing q , particularly to $q = 0$ or $q = -1$, will reduce this expression further.

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APPENDIX A. EXAMPLE

We present an example of Theorem 4.1 by computing the term obtained from a particular GT pattern. We take $\lambda = (1, 0, 0)$, so $\lambda + \rho = (3, 1, 0)$, and the GT pattern:

$$\begin{array}{ccc} 3 & 1 & 0 \\ & 2 & 0 \\ & & 1 \end{array}$$

Having fixed a particular $T \in SG(3, 1, 0)$, we iterate i , starting with $i = 1$. Then we have the set $\Omega(r_2) = \{\text{Id}, [1\ 2]\}$. We denote $[1\ 2]$ as ϕ . Then $\phi(r_2) = (1, 1)$ and $l(\phi) = 1$.

We thus have two possibilities to consider: one for each raising operator. In each case, we display the relevant patterns and compute the determinant of M . To minimize confusion, we have subscripts for integers that appear multiple times in a pattern.

The raising operator Id .

$$\begin{array}{ccc} 3 & 1 & 0_1 \\ & 2 & 0_2 \end{array}$$

This gives the matrix and coefficient:

$$(A.1) \quad \left| \begin{pmatrix} w(2) & 1 \\ d(1) & w(0_2) \end{pmatrix} \right| = \begin{vmatrix} 1-q & 1 \\ -qt^2 & 1+t \end{vmatrix} = 1 - q + t - qt + qt^2.$$

The raising operator ϕ .

$$\begin{array}{ccc} 3 & 1_1 & 0 \\ & 1_2 & 1_3 \end{array}$$

As $l(\phi_1) = 1$ we multiply the determinant of the associated matrix by t^1 to obtain

$$(A.2) \quad t \cdot \left| \begin{pmatrix} w(1_2) & 1 \\ d(1_1) & w(1_3) \end{pmatrix} \right| = t \cdot \begin{vmatrix} 1 & 1 \\ -q & -q - qt \end{vmatrix} = -qt^2.$$

That concludes the consideration of possible second rows. We add all of the coefficients from each case of a second row, i.e. (A.1) and (A.2), to find the total coefficient of

$$(A.3) \quad 1 - q + t - qt + qt^2 - qt^2 = (1 - q)(1 + t).$$

Now iterating to $i = 2$, we have that $\Omega(r_3)$ contains only the identity.

$$\begin{array}{ccc} & & 0 \\ 2 & & \end{array}$$

The matrix is 1 by 1, so its determinant is just

$$(A.4) \quad |w(1)| = (1 - q).$$

Finally, we take the product of all the coefficients we got from each row, i.e. (A.3) and (A.4), and multiply this with $x^{m(T)}$ where $m(T) = (2, 1, 1)$. Thus, by Theorem 4.1, the GT pattern contributes the monomial

$$(A.5) \quad (1 - q)^2(1 + t)x_1^2x_2x_3,$$

to the summation. As this is the unique GT pattern with top row $\lambda + \rho = (2, 1, 0)$ and $m(T) = (2, 1, 1)$, we should find that (A.5) gives the coefficient of $x_1^2x_2x_3$ in the expansion of $\Delta_3(q) \cdot R_{(1,0,0)}(x; t)$. The reader can verify this is indeed the case.